BELLCOMM, INC.

SUBJECT: Instantaneous Impact Point Analysis - Case 218

DATE: July 13, 1966

FROM: R. Y. Pei

ABSTRACT

The instantaneous impact point (IIP) corresponding to a given set of thrust cut-off conditions is analyzed. Analytical solutions are obtained for various impact parameters. Several graphical solutions are formulated for the purpose of generating such information for mission planning purposes. A new form of graph paper with trigonometric scales is introduced.

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MEMORANDUM FOR FILE

Prediction of the flight profile of a mission in case of thrust failure prior to attainment of orbital speed is of importance to range safety considerations. Mission planners require such information as the loci of sub-vehicular points (Ground Track) as well as the instantaneous impact point (IIP Trace)

When the trajectory of a proposed mission is actually simulated on a computer,* it is feasible to incorporate a routine that will yield the instantaneous impact point resulting from the premature termination of thrust at a given point during a powered phase of the flight. On the other hand, if a parametric study is to be conducted in order to establish some guidelines in tradeoffs between performance and range safety, it would be desirable to generate such information quickly without having to resort to the computer.

The purpose of this paper is to present graphical data based on analytical formulae derived for this purpose and to suggest a few examples where such data may be used with advantage. A ballistic model has been used for simplicity and the effect of earth's rotation has been neglected.

The range angle, i.e., the geo-central angle subtended by the vehicle position vectors at cut-off and impact respectively, is given by the following:**

$$\Delta\theta = \left[\arc\cos\left(\frac{\xi - 1}{\sqrt{1 - \xi\eta}}\right)\right]_{\text{Cut-off}}^{\text{Impact}} \qquad (1)$$

where

$$\Delta\theta = \text{Range Angle}$$

$$\xi = \nu^2 \cos^2 \gamma \qquad (2)$$

[redacted]

*For example, the Bellcomm Apollo Simulation Program (ASP).

**Formulae used in this paper are derived in the Appendix. Definitions of terms are grouped under section entitled "Nomenclature" at the end of paper.

$$\eta = 2 - \nu^2 \tag{3}$$

ν = energy index

$$= \frac{\text{cut-off velocity}}{\text{circular velocity at cut-off altitude}}$$

γ = flight path angle relative to local horizon.

In order to apply Equation (1), the variables ξ, η are computed at cut-off point, using the cut-off velocity, altitude and flight path angle. The values of these same variables at the point of impact are computed directly from the following relationships:

$$\xi_i = \rho \xi_{co} \tag{4}$$

$$\eta_i = \frac{1}{\rho} \eta_{co} \tag{5}$$

where the subscripts i and co denote impact and cut-off respectively, and

$$\rho = \frac{r_{co}}{r_{oo}} \tag{6}$$

r being the magnitude of the position vector.

By virtue of Lambert's theorem, the impact time normalized with respect to the circular period at cut-off altitude is given by the following approximation:

$$2\pi \frac{t}{T_{co}} = \frac{1}{\eta^{3/2}} \left[(\alpha - \sin\alpha) - (\beta - \sin\beta) \right] \tag{7}$$

where

T_{co} = circular period at cut-off altitude

$$\sin \frac{\alpha}{2} = \sqrt{\frac{\eta}{2} \left(1 + \sin \frac{\Delta\theta}{2} \right)} \tag{8}$$

$$\sin \frac{\beta}{2} = \sqrt{\frac{\eta}{2} \left(1 - \sin \frac{\Delta\theta}{2} \right)} \tag{9}$$

The approximation is valid when the cut-off altitude above ground is small as compared to the magnitude of the position vector. The effect of a finite altitude at cut-off may be accounted for with a correction in the values of the variables η and $\Delta\theta$ as follows:

$$\delta(\Delta\theta) = 2 \sin^{-1} \frac{\sqrt{1+\rho^2-2\rho\cos\Delta\theta}}{1+\rho} - \Delta\theta \qquad (10)$$

$$\sim 0$$

$$\delta(\eta) = \frac{1}{2} \frac{\eta}{\rho} (1+\rho) - \eta \qquad (11)$$

$$\sim -\frac{1}{2} \varepsilon\eta$$

where

$$\varepsilon = \frac{r_{co} - r_{oo}}{r_{oo}}$$

The latitude and longitude of the instantaneous impact point are related to those of the sub-vehicular point at cut-off by the following formulae:

$$\sin L_i = \sin L_{co} \cos(\Delta\theta) + \cos L_{co} \sin(\Delta\theta) \cos \beta_{co} \qquad (12)$$

$$\sin(\lambda_i - \lambda_{co}) = \frac{\sin(\Delta\theta) \sin \beta_{co}}{\cos L_i} \qquad (13)$$

where

L = latitude

λ = longitude

β = azimuthal angle

and $\Delta\theta$ and the subscripts i and co retain their definitions as before. Derivations of the above formulae are included in the Appendix.

Method of Solution

In this paper, graphical solutions are proposed for the problems formulated above. These solutions have been devised for the purpose of obtaining answers rapidly and with an accuracy useful for mission planning purposes.

1. Solution to the Range Angle Problem - Consider the denominator in the argument of the inverse cosine function in Equation (1). It is obvious from Equations (4) and (5) that this quantity remains constant for both the cut-off and the impact points. On a graph paper with polar coordinates, an arc is drawn with radius equal to $\sqrt{1-\xi\eta}$. Along any radius, mark off quantities equal to $(\xi-1)$ and $(\rho\xi-1)$ respectively, from the origin and draw straight lines at these points perpendicular to the radius chosen, intersecting the arc at two points. Radii joining the origin to these points enclose the range angle. As an example, the following is given:

$$\nu_{co}^2 = 0.7$$

$$\gamma_{co} = 20°$$

$$\rho = 1.1$$

We obtain:

$$\xi = 0.615$$

$$\eta = 1.3$$

$$\sqrt{1-\xi\eta} = 0.447$$

$$\xi-1 = -0.385$$

$$\rho\xi-1 = -0.324$$

The proposed graphical solution is shown in Figure 5 and the range angle is obtained to be 13°. Note that in accordance with the sign of the quantities $(\xi-1)$ and $(\rho\xi-1)$, the radius corresponding to $\theta=\pi$ has been chosen. In this manner, the solution yields not only the range angle, but also the true anomalies. When ρ

is near unity, that is when the altitude at cut-off is exceedingly small, it will be difficult to realize a satisfactory resolution between the quantities $(\xi-1)$ and $(\rho\xi-1)$ in a convenient graphical solution. In this case, the well known simplified solution corresponding to a flat earth should perhaps suffice.

2. Solution to the Impact Time Problem - Equations (8) and (9) yield the following expressions:

$$\eta = \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} \qquad (14)$$

$$\sin \frac{\Delta\theta}{2} = \frac{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2}}{\sin^2 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2}} \qquad (15)$$

Consequently if $\sin\frac{\alpha}{2}$ and $\sin\frac{\beta}{2}$ are chosen as coordinates, Equation (14) is that for a circle of radius $\sqrt{\eta}$ while Equation (15) represents families of straight lines in the real plane with slopes of

$$\pm \sqrt{\frac{1 - \sin(\Delta\theta/2)}{1 + \sin(\Delta\theta/2)}} \quad .$$

For $\pi \geq \alpha \geq \beta \geq 0$, only the positive quadrant is chosen. Thus, to a set of values of η and $\Delta\theta$, there correspond a circle and a straight line. Their intersection determines the values of $\sin\frac{\alpha}{2}$ and $\sin \frac{\beta}{2}$. Solution to Equation (7) is obtained readily with the aid of linear scale and a sine scale. This is illustrated in Figure 6. The readings on the abscissa are doubled and transferred to the scales for the solution to the impact time problem.

Solution (1) and (2) render it desirable to work with the variables ξ and η introduced in this paper.

3. Solution to the IIP Problem - The purpose of the constructions shown in Figures 2,3, and 4 is three fold, viz: (i) to determine the instantaneous impact point for a set of cut-off conditions, (ii) to study the effect of a yaw program on the IIP trace and (iii) to introduce the use of a Trigonometric Paper to solve transcendental equations. Equation (12) shows that if the abscissa and ordinate are scaled according to the cosine and sine functions respectively, then for a given set of cut-off conditions, viz: (i) the latitude of the cut-off point and (ii) the range angle as obtained above, then the equation for the impact point is a straight line in terms of the azimuthal angle.

This line is easily determined by considering the value of L_i for $\beta_{co}=0$ and $\beta_{co}=\pi$ respectively.

$$\sin L_i = \sin(L_{co}+\Delta\theta) \quad \text{for} \quad \beta_{co}=0$$

$$\sin L_i = \sin(L_{co}-\Delta\theta) \quad \text{for} \quad \beta_{co}=\pi$$

In order to study the longitude of the impact point, we note that Equation (13) may be written as:

$$\frac{\sin^2 \Delta\theta}{\kappa^2} + \sin^2 L_i = 1 \tag{16}$$

where

$$\kappa = \frac{\sin(\lambda_i-\lambda_{co})}{\sin \beta_{co}} \tag{17}$$

Equation (16) is that of an ellipse, while Equation (17) is a straight line. These are plotted in Figure 3 for various values of κ. The impact point longitude may thus be quickly determined by entering Figure 3 with the range angle and the impact latitude determined earlier and locating the ellipse on which this point lies. The κ - value of the ellipse is then used to locate the straight line which shows the dependence of the impact longitude on a yaw program. Figure 4 shows two such cases.

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Attachment
Appendix
Figure 1 - 6

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Nomenclature

a	Semi-major axis of ellipse
c	Chord joining two points on ellipse
e	Eccentricity
L	Latitude
r	Focal radius
s	Semi-perimeter defined in Eq. (A-12')
t	Time elapsed between two points on ellipse
T	Period
v	Velocity
α	Variable defined in Eq. (A-12')
β	Variable defined in Eq. (A-12')
γ	Flight path angle
ε	Error
η	Variable defined in Eq. (3)
θ	True anomaly
κ	Parameter defined in Eq. (17)
λ	Longitude
μ	Gravitational constant
ν	Energy index defined in Eq. (A-3)
ξ	Variable defined in Eq. (2)
ρ	Ratio of focal radii defined in Eq. (6)

Subscripts:
- co : Cut-off
- i : Impact
- oo : Used to designate average earth radius.

APPENDIX

1. Impact Range Angle - After cut-off the vehicle is assumed to follow an elliptic trajectory, thus neglecting the effect of atmospheric drag.* The magnitude of velocity is:

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{A-1}$$

The flight path angle is:

$$\cos \gamma = \frac{\sqrt{\mu a(1-e^2)}}{rv} \tag{A-2}$$

Solving for e between (A-1) and (A-2) and letting $\nu^2 = \frac{v^2}{\mu/r}$, one obtains

$$e = \sqrt{1 - \nu^2(2-\nu^2) \cos^2 \gamma} \tag{A-3}$$

The true anomaly is given by:

$$\cos \theta = \frac{a(1-e^2) - r}{er} \tag{A-4}$$

Noting Equations (A-2) and (A-3), (A-4) may be written as:

$$\cos \theta = \frac{\nu^2 \cos^2 \gamma - 1}{\sqrt{1 - (2-\nu^2)\nu^2 \cos^2\gamma}}$$

$$= \frac{\xi - 1}{\sqrt{1 - \xi\eta}} \tag{A-5}$$

where

$$\xi = \nu^2 \cos^2 \gamma$$

$$\eta = 2 - \nu^2$$

From Equation (A-1), it follows that the energy index ν^2 may vary between 0 and 2. The impact range angle may be evaluated from Equation (A-5) by computing the values of ξ and η, and consequently the true anomalies at cut-off and impact and taking the difference. These are known at the cut-off point. At the impact point, it is required that the radius vector equals the earth's radius r_{oo}. Thus,

$$v_i^2 = \mu\left(\frac{2}{r_{oo}} - \frac{1}{a}\right) \tag{A-6}$$

*See Sec. 4 for a discussion of this effect.

Subtracting from it the velocity squared at cut-off point,

$$v_i^2 = v_{co}^2 + 2\mu\left(\frac{1}{r_{oo}} - \frac{1}{r_{co}}\right) \qquad \text{(A-7)}$$

It follows, therefore, that

$$\nu_i^2 = \nu_{co}^2 \frac{r_{oo}}{r_{co}} + 2\left(1 - \frac{r_{oo}}{r_{co}}\right) \qquad \text{(A-8)}$$

and,

$$\eta_i = \frac{1}{\rho}\,\eta_{co}$$

where

$$\rho = \frac{r_{co}}{r_{oo}}$$

From Equation (A-2), it follows that

$$r\nu^2 \cos^2 \gamma = a(1-e^2) \qquad \text{(A-9)}$$

Since the right hand side is constant for a given ellipse, one may write

$$\xi_i = \rho\xi_{co} \qquad \text{(A-10)}$$

2. Time of Impact - Let the impact range angle evaluated with the help of Equations (A-5), (A-8), and (A-10) be denoted by $\Delta\theta$. The length of the chord joining the cut-off and the impact points is then

$$c = r_{oo}\sqrt{1 + \rho^2 - 2\rho\cos(\Delta\theta)} \qquad \text{(A-11)}$$

Applying Lambert's Theorem, the time interval between cut-off and impact is given by:

$$\sqrt{\mu}\,t = \sqrt{a^3}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \qquad \text{(A-12)}$$

where,

$$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$$

$$\sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}} \qquad \text{(A-12')}$$

$$2s = r_{oo} + r_{co} + c$$

$$a = r/\eta$$

Substituting Equation (A-11) for the value of c,

$$\sin^2\frac{\alpha}{2} = \frac{s}{2a} = \frac{\eta_{co}}{4\rho}\left[1 + \rho + \sqrt{1+\rho^2-2\rho\cos\Delta\theta}\right] \qquad \text{(A-13)}$$

$$\sin^2\frac{\beta}{2} = \frac{s-c}{2a} = \frac{\eta_{co}}{4\rho}\left[1 + \rho - \sqrt{1+\rho^2-2\rho\cos\Delta\theta}\right] \qquad \text{(A-14)}$$

When the altitude at cut-off is neglected, ρ becomes unity and Equations (A-13) and (A-14) simplify to:

$$\frac{s}{2a} \sim \frac{\eta_{co}}{2}\left(1 + \sin\frac{\Delta\theta}{2}\right) \qquad \text{(A-15)}$$

$$\frac{s-c}{2a} \sim \frac{\eta_{co}}{2}\left(1 - \sin\frac{\Delta\theta}{2}\right) \qquad \text{(A-16)}$$

In order to account for the effect of the altitude, the values from η_{co} and $\Delta\theta$ must be corrected. Let $\delta\eta_{co}$ and $\delta(\Delta\theta)$ be such corrections. Then Equation (A-13) through (A-16) demand,

$$\frac{\eta_{co} + \delta\eta_{co}}{2}\left[1 + \sin\frac{(\Delta\theta + \delta\Delta\theta)}{2}\right] =$$

$$= \frac{\eta_{co}}{4\rho}\left[1 + \rho + \sqrt{1+\rho^2-2\rho\cos\Delta\theta}\right] \qquad \text{(A-17)}$$

$$\frac{\eta_{co} + \delta\eta_{co}}{2}\left[1 - \sin\frac{(\Delta\theta + \delta\Delta\theta)}{2}\right] =$$

$$= \frac{\eta_{co}}{4\rho}\left[1 + \rho - \sqrt{1+\rho^2-2\rho\cos\Delta\theta}\right] \qquad \text{(A-18)}$$

Solving for $(\eta_{co} + \delta\lambda_{co})$ and $(\Delta\theta + \delta(\Delta\theta))$, the corrected values of these variables become

$$\eta_{co} + \delta\eta_{co} = \frac{1}{2}\frac{\eta_{co}}{\rho}(1 + \rho) \qquad \text{(A-19)}$$

$$\sin\left(\frac{\Delta\theta + \delta\Delta\theta}{2}\right) = \frac{\sqrt{1 + \rho^2 - 2\rho\cos\Delta\theta}}{1 + \rho} \qquad \text{(A-20)}$$

The normalized impact time is obtained from Equation (A-12) as follows:

$$2\pi\frac{t}{T_{co}} = \frac{1}{\eta^{3/2}}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \qquad \text{(A-21)}$$

It is to be noted that if the corrected values in Equation (A-19) and (A-20) are to be used for η and $\Delta\theta$, then the impact time computed according to Equation (A-21) must be multiplied by

$$\left[(1 + \rho)/2\rho\right]^{3/2}.$$

As a further simplification, if the altitude is small such that ρ may be replaced by $(1+\varepsilon)$, then a first order approximation yields,

$$\delta\eta_{co} \sim -\frac{1}{2}\varepsilon\eta \qquad \text{(A-22)}$$

$$\delta(\Delta\theta) \sim 0 \qquad \text{(A-23)}$$

for small $\Delta\theta$ while the resultant impact time from Equation (A-21) will have to be multiplied by $\left|1 - \frac{3}{4}\varepsilon\right|$. In case $\Delta\theta$ approaches π, then Equation (A-23) must be modified by taking second order terms into account:

$$\delta(\Delta\theta) \sim -\frac{1}{4}\varepsilon^2 \tan\frac{\Delta\theta}{2} \tag{A-24}$$

3. Instantaneous Impact Point - The position of the instantaneous impact point is determined by solving the spherical triangle in Figure 1. Thus,

$$\sin L_i = \sin L_{co} \cos(\Delta\theta) + \cos L_{co} \sin(\Delta\theta) \cos \beta_{co} \tag{A-25}$$

where β_{co} is the azimuthal angle at cut-off. Also, the law of sines yields,

$$\frac{\cos L_{co}}{\sin \beta_i} = \frac{\sin(\Delta\theta)}{\sin(\lambda_i - \lambda_{co})} = \frac{\cos L_i}{\sin \beta_{co}} \tag{A-26}$$

Solving for $\sin(\lambda_i - \lambda_{co})$, the following is obtained,

$$\sin(\lambda_i - \lambda_{co}) = \frac{\sin(\Delta\theta) \sin \beta_{co}}{\cos L_i} \tag{A-27}$$

where L_i has been determined by Equation (A-26).

4. Effect of Atmospheric Drag - The atmospheric drag will reduce the Impact Range Angle by an amount depending on the vehicle ballistic number, atmospheric data and cut-off conditions. This reduction has been estimated by comparing the ballistic results of the given method to those of integrated trajectories using the ICAO atmospheric data and a ballistic number of 0.32 ft/slug. The energy index used range from ν=0.7 to ν=0.95 with flight path angles of 0°, 5° and 10° respectively. The comparison shows that the ballistic result overestimates the range angle by approximately 10 n.m. or 5% (ν=0.7, γ=10°) to 200 n.m. or 20% (ν=0.9, γ=0°). These deviations may be compared with the range of a standard landing zone.

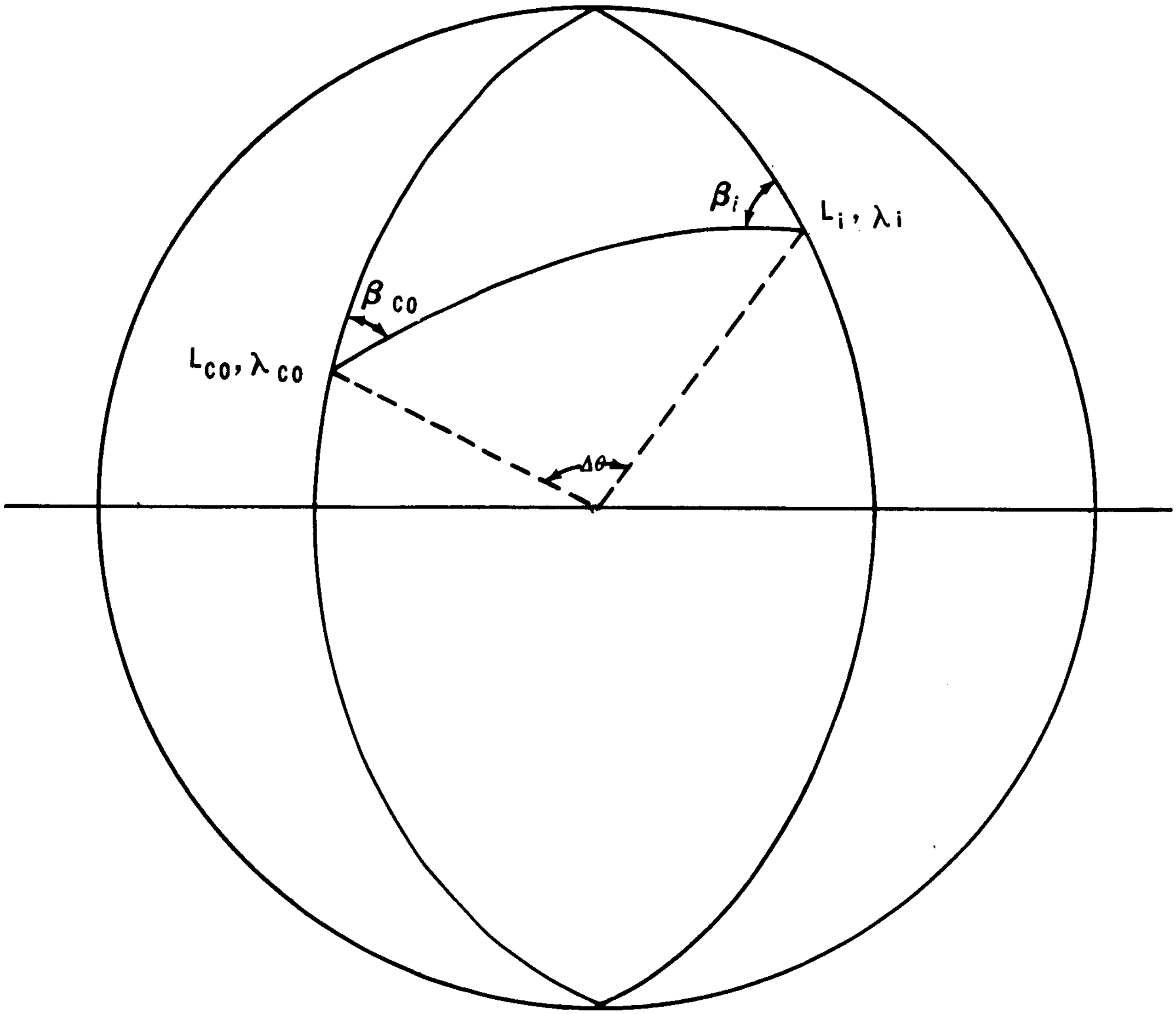

FIGURE I

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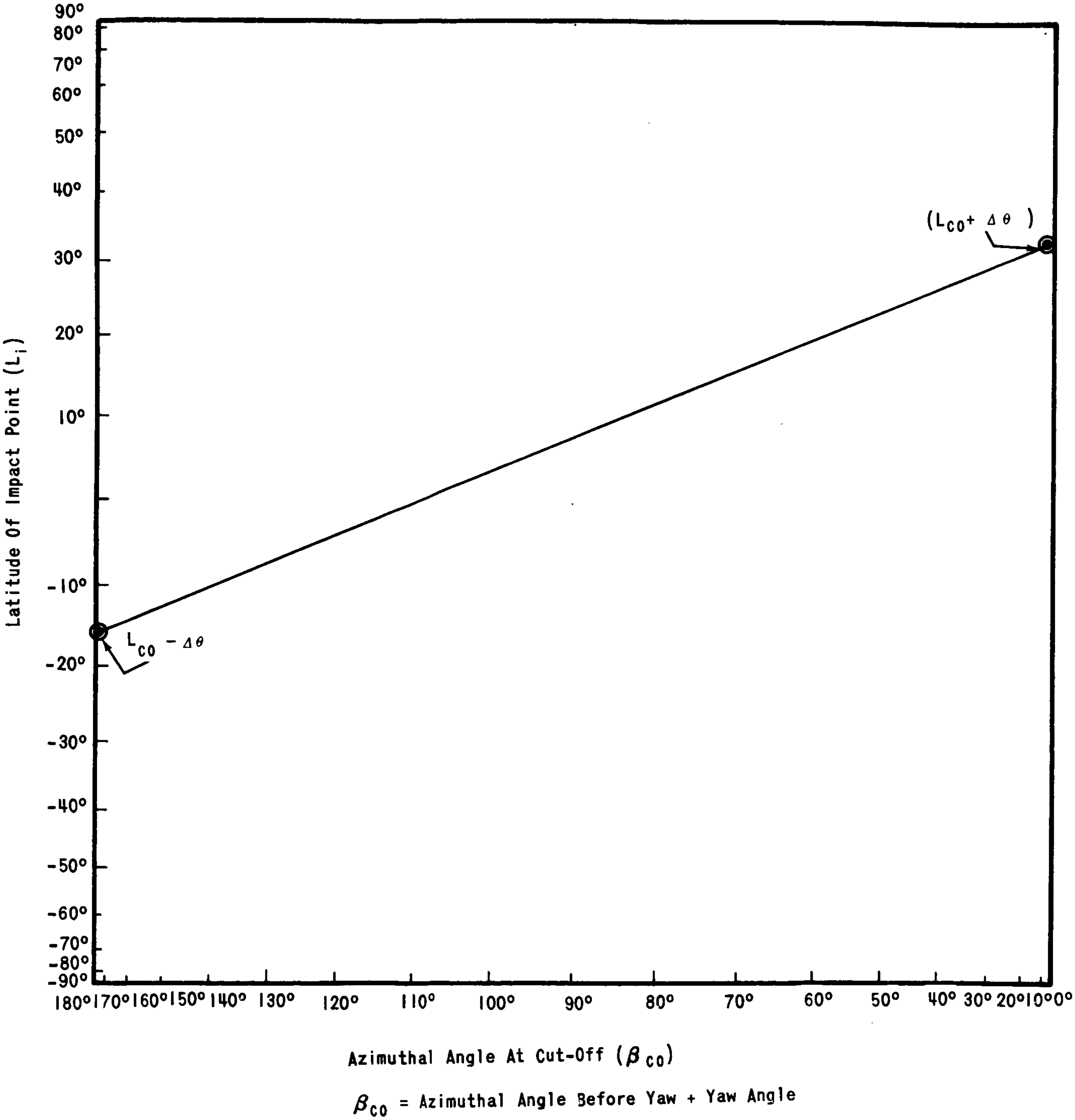

FIGURE 2 - EFFECT OF YAW ON IIP LATITUDES

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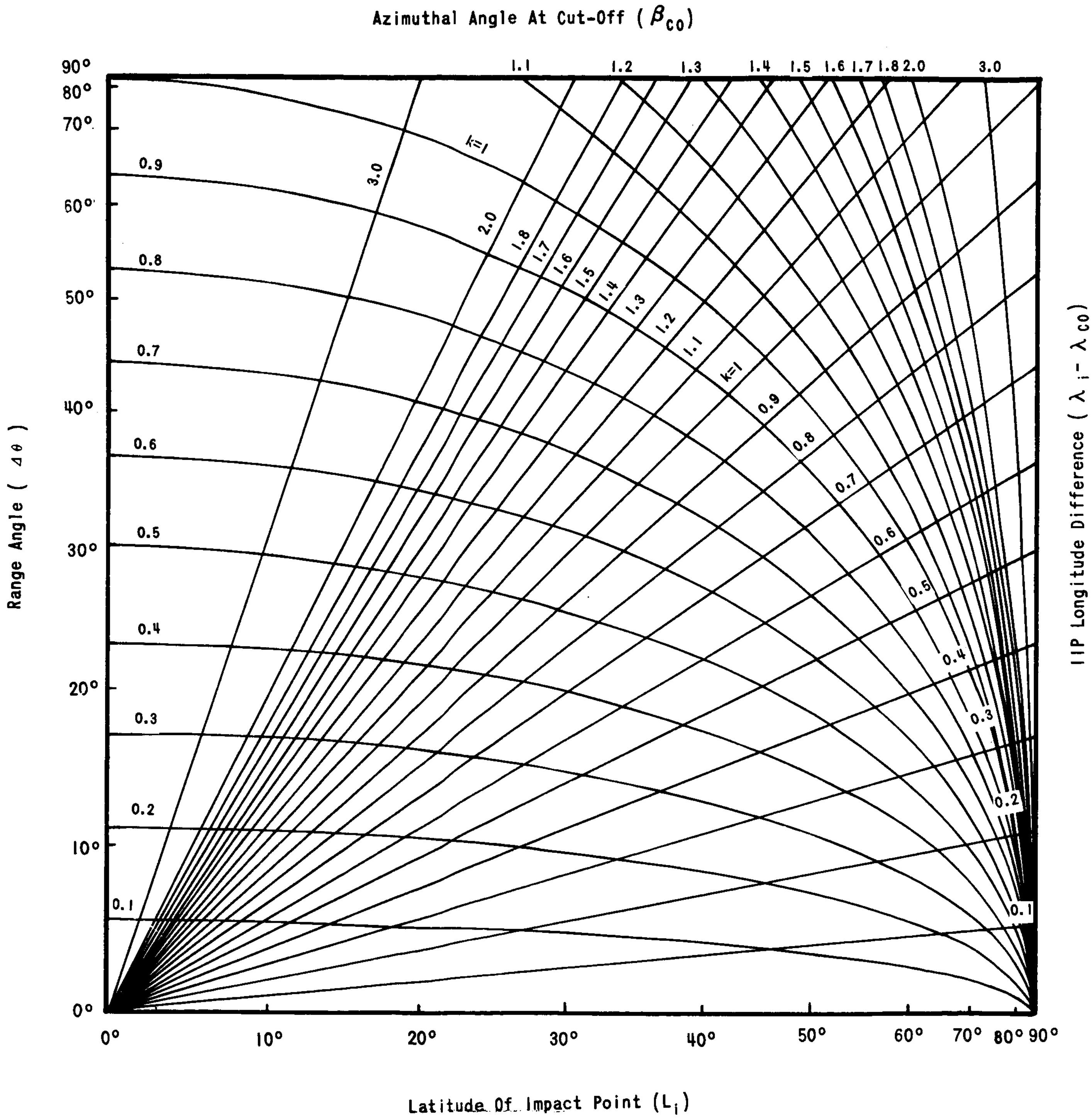

FIGURE 3 - EFFECT OF YAW ON IIP LONGITUDES

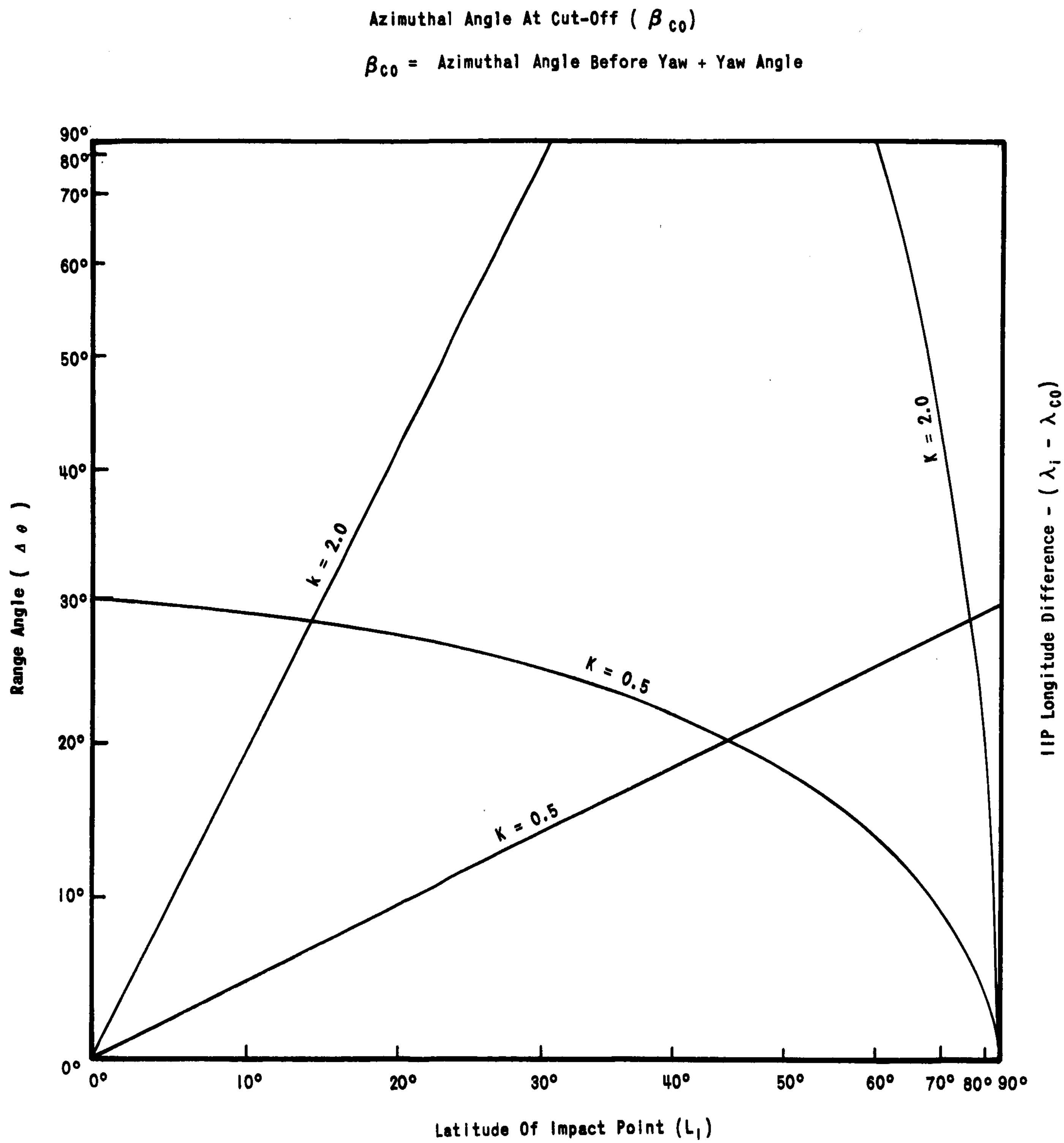

FIGURE 4 - EFFECT OF YAW ON IIP LONGITUDES

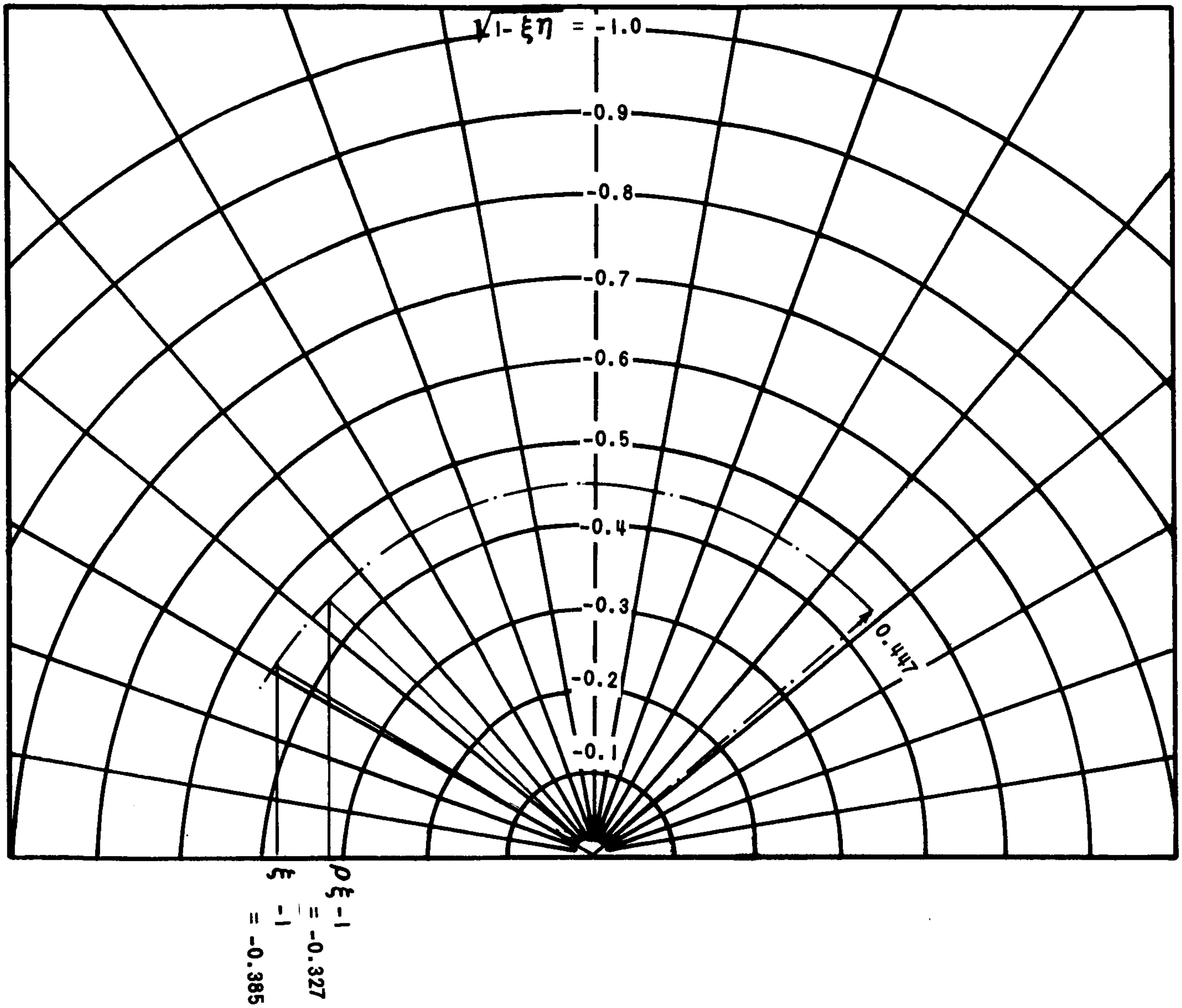

FIGURE 5 - SOLUTION TO RANGE ANGLE PROBLEM

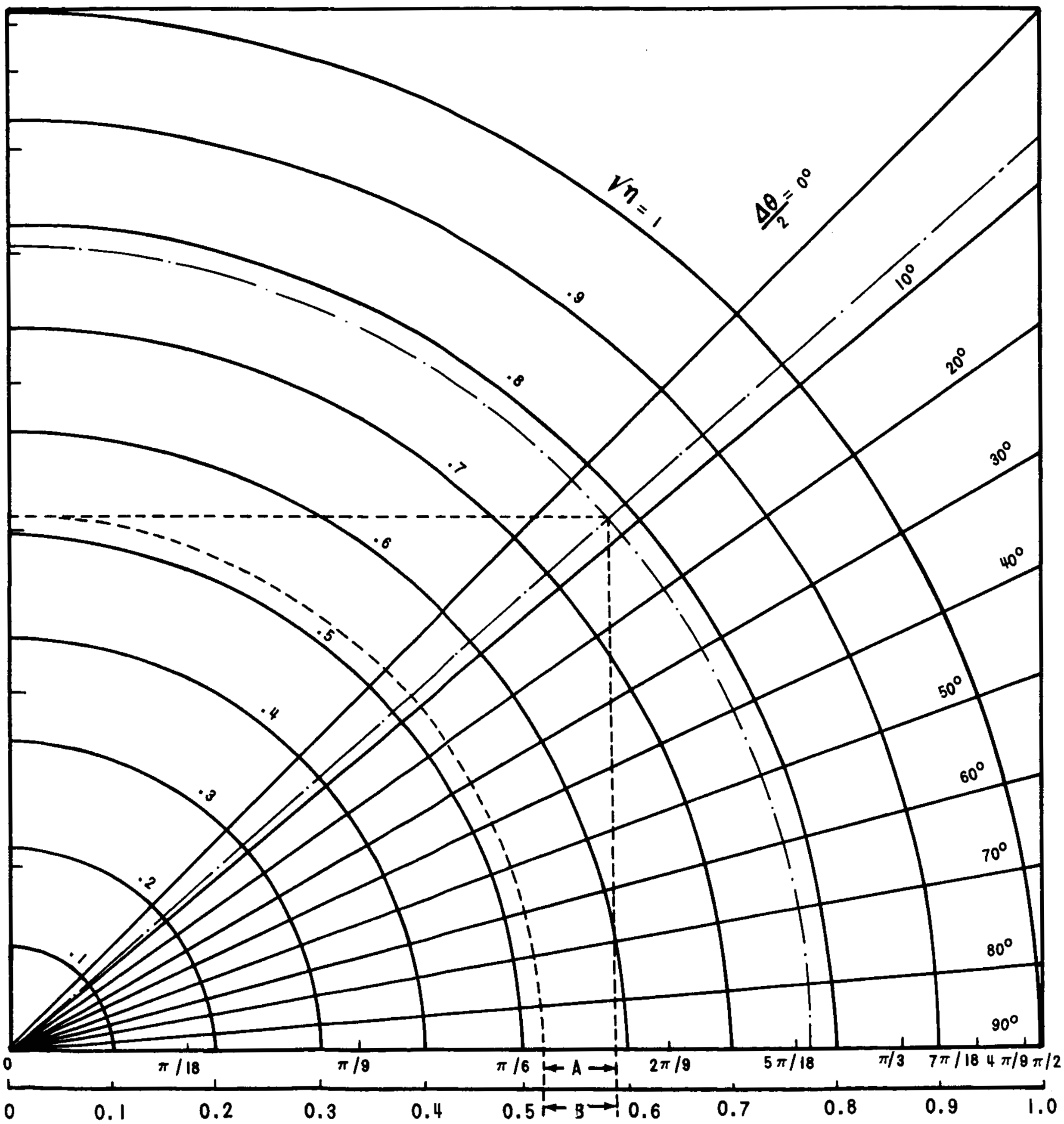

FIGURE 6 - SOLUTION TO IMPACT TIME PROBLEM